Research Again On the Cutting Plane Method Resolving ILP Problems

RECHERCHE SUR LA METHODE DE COUPE PLANE DANS LA RESOLUTION DES PROBLEMES ILP

Xiong Yijie[1] **Ren Jing**[2]

Abstract: How to resolve ILP problems is all along hotspot subject In the Operation Research region. The author of the paper, by the demonstration research method, analyzed the errors of Cutting Plane Method used in resolving ILP, and put forth a new principle, i.e. "it is such as a cutting plane equation that has more great restriction on a given problem". At the same time, the author pointed out that there are two problems that would be noticed in using course. The paper has important theory and practice value.
Key words: Integer Linear Programming (ILP), Cutting plane equation, Export Equation

Résumé: Comment résoudre les problèmes ILP est toujours un sujet chaud dans le milieu de la Recherche d'Opération. L'auteur de cet essai, à travers la méthode de démonstration, a analysé les fautes de la Méthode de Coupe Plane utilisée pour résoudre ILP et a proposé un nouveau principe, par exemple : « il est comme une équation de coupe plane qui a plus de restrictions sur un problème donné. ». En même temps, l'auteur indique qu'il y a deux problèmes qui seraient notés au cours de l'utilisation. Cet article revêtit une valeur importante théorique et pratique.
Mots-Clés: ILP(Integer Linear Programming /programmation linéaire du nombre entier), équation de coupe plane, équation d'exportation

Integer Linear Programming (ILP is the abbreviation of the Integer Linear Programming) is a very essential type in mathematical programming. It had a wide applied foreground in economic life. However, how to solve this kind of problems is the focus in study field, it is necessary to resolve many correlative problems in the further research. For example, the problem that uses cutting plane method to solve ILP is one of them, due to the cutting plane equation isn't unique, how to find an effective and quickly convergent ILP equation hasn't still been solved.

1. DEFINITION OF CUTTING PLANE EQUATION

The optimum simplex table that solves the Slack Linear Programming problem is the foundation to solve the cutting plane equation. Establishment of the equation needs to extract out a exporting equation from the optimum simplex table. The exporting equation may be arbitrary one of restrict equations. So it is very important problem to choice the restrict equation as the exporting equation that forming the shear restriction so as to find the integer solution. See an example below.

【Example 1】 Use to cutting equation method to solve a following integral programming

$$Max.Z = 8x_1 + 5x_2$$

s.t.

$$\begin{cases} 2x_1 + 3x_2 \le 12 \\ 2x_1 - x_2 \le 6 \\ x_1, x_2 \ge 0 \quad and \quad x_1, x_2 \ as \ integer \end{cases}$$

【Solution】In the general , using the cutting plane method to solve the equation, demands three steps:

The first step: The optimum simplex table that solves the Slack Linear Programming problem is obtained using simplex method as table 1:

[1] Business Administration School, Xi'an University of Technology, China.
[2] Business Administration School, Xi'an University of Technology, China.
*Received 26 May 2006 ; accepted 24 July 2006

Table 1 the optimal simplex table of the SLP problem in example 1

		x_1	x_2	x_3	x_4
Z	-37.5	0	0	-2.25	-1.75
x_2	1.5	0	1	0.25	-0.25
x_1	3.75	1	0	0.125	0.375

The second step: solving cutting plane equation. The cutting plane equation isn't unique. For example, it can as the export equation with the row that include the variable x_1, also can as the export equation with the rows that include the variable x_2. Here we use the row that includes the variable x_2 as the export equation. Thus the restriction can be expressed as follows:

$$x_1 + 0.125x_3 + 0.375x_4 = 3.75$$

All the no integral number parameter and constant will be expressed using integral fraction and a pure decimal fraction, then the restriction can rewrite as:

$$x_1 + (0 + 0.125)x_3 + (0 + 0.375)x_4 = 3 + 0.75$$

While decomposing the variable the coefficient that changes quantity, it is need to solve in two cases: positive coefficient and negative. The first case is decomposing the coefficient directly, the second case is using the complement, for example, if selecting the first restriction above, it can resolve for:

$$x_2 + (0 + 0.25)x_3 - (1 - 0.75)x_4 = 1 + 0.5$$

Then, all integral fractions move the left side of the equation, and all pure decimal fractions move the right side of the equation, then it can be got as follows:

$$x_1 - 3 = 0.75 - (0.125x_3 + 0.375x_4)$$

Obviously, if the value of variable x_1 is integer, the left of the equation should be integer, and also the right of the equation should be integer. But according to the restriction condition: if the variable is great and equal zero, it is positive in the right parentheses and also it is not possible between 0 and 0.75, hence the cutting plane equation or the cutting restriction is

$0.75 - 0.125x_3 - 0.375x_4 \le 0$ （the cutting restriction equivalences $x_1 - 3 \le 0$）

This cutting plane equation can be expressed as the Fig. 1. That is the cutting plane equation has cut the part of feasible region.

The third step: the cutting plane equation is put into the simplex table in the first step, the P_1 can be obtained (the optimum simplex table of SLP is P_0). The last slack problem can be solved using the simplex method of duality. The key to improve solving efficiency of the step is selected a fit cutting plane equation in the second step. Thus how to select the cutting plane equation and export equation is the core step using the cutting plane equation to solve the problem.

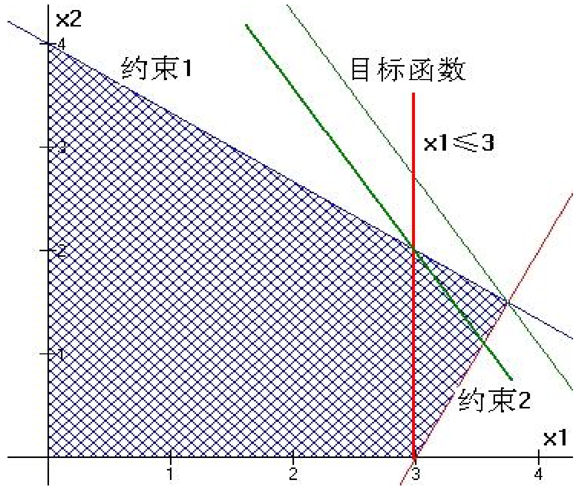

Fig.1 sketch map of the cutting plane equation

2. REVIEW THE STUDY

The selecting principle or standard of the export equation cannot be seen in many textbooks that introduced the cutting plane method.

In 1985, Yusen Yu expatiated a principle as the Eq. 7 in his 《mathematics programming principle and method》.This principle can be expressed as:

The superior solution that the SLP problem gets is:）

$$x_i = [b_i] + f_i$$

Where: $[b_i]$ express the maximum integer value of x_i, and f_i express the pure decimal part of x_i, then the cutting plane equation should select f_i greater variable corresponding equation as export equation in solution combination.[3]

In September 1996, this principle is recommended by Jiequn Tan in 《the integral programming cutting plane method to disintegrate to explore lately》.[4]June in 1998, Yunquan Hu pointed out: “it can improve the cutting efficiency and decrease the cutting times using the method that select the row having the pure decimal part of the maximum decimal to set the cutting plane restriction” in << Operation Research Tutorial >>, [5]and

[3] Yu Yusen: "Mathematical programming Principle and Method", Central China engineering institute publishing company, in 1985 version.

[4] Tan Jiqun: "Integer programming Cutting plane algorithm Disintegration Searches newly", "Guangxi Teachers' college Journal (Natural sciences Version)" 1996Vol.13No.3.

[5] Hu Yunquan and Guo Yaohuang: "Operations research Course", Qinghua University publishing company, in June, 1998 1st edition.

in the edition 2 of the book, the author affirmed the principle.

This principle is also named the Yushi principle, and it is fit example 1 above. Using the principle, the row including x_1 is as the export equation, and it can get the optimism integer solution through one time iterative operation. The process is described in Table 2.

The Yushi(俞氏) Principle also applies to other examples. However, the writers discovered, still the principle has exception, such as the example 2:

Table 2 Example 1 joins to a pure form calculation table after cutting constraint

		x_1	x_2	x_3	x_4	x_5
z	-37.5	0	0	-2.25	-1.75	0
x_2	1.5	0	1	0.25	-0.25	0
x_1	3.75	1	0	0.125	0.375	0
x_5	-0.75	0	0	-0.125	-0.375	1
z	-34	0	0	-5/3	0	-14/3
x_2	2	0	1	1/3	0	-2/3
x_1	3	1	0	0	0	1
x_4	2	0	0	1/3	1	-8/3

【Example 2】 Solving following integral programming using the cutting plane method

$$Min.Z = 6x_1 + 8x_2$$

$$\text{s.t.} \begin{cases} 3x_1 + x_2 \geq 4 \\ x_1 + 2x_2 \geq 4 \\ x_1, x_2 \geq 0 \quad and \quad x_1, x_2 \ as integer \end{cases}$$

To solve the SLP problem of the ILP, and the optimum simplex table can be obtained as Table 3

Table 3 SLP problem the optimum simplex table of SLP problem in Example 2

		X1	**X2**	**X3**	**X4**
Z	88/5	0	0	-4/5	-18/5
X1	4/5	1	0	-2/5	1/5
X2	8/5	0	1	1/5	-3/5

Obviously, according to the principle, the row of including x_1 can be as the export equation, but actually the cutting restriction by the equation is:

$$x_1 - x_3 = \frac{4}{5} - (\frac{3}{5}x_3 + \frac{1}{5}x_4) \leq 0$$

It can't get optimum solution through one time iterate operation. But according to the row including x2 is as the export equation and the cutting constraint:

$$x_2 - x_4 - 1 = \frac{3}{5} - (\frac{1}{5}x_3 + \frac{2}{5}x_4) \leq 0$$

It also can get optimum integral solutions by one iterate operation and the result is shown in Table 4:

Table 4 Simplex calculation table after cutting constraint in example 2

		X1	X2	X3	X4	X5
Z	88/5	0	0	-4/5	-18/5	0
X1	4/5	1	0	-2/5	1/5	0
X2	8/5	0	1	1/5	-3/5	0
X5	-3/5	0	0	(-1/5)	-2/5	1
Z	20	0	0	0	-2	-4
X1	2	1	0	0	1	-2
X2	1	0	1	0	-1	1
X3	3	0	0	1	2	-5

Once the author found other examples of inapplicability in the application. In addition, in the case that the pure decimal part of the variables is equal, the Yushi（俞氏）principle is also unfit. Whereas the writer pointed out a problem in a paper published by《strategy and management》 in 2003, the problem is that choose the restriction equation of the minimum absolute value as the export equation, the minimum absolute value comes from the pure decimal of positive coefficient divided by relevant verify data in restriction equation. The principle can be named as the exclusivity principle of export equations. [6]If the principle is not met, perhaps the feasible integer solution can be obtained, it is unnecessary the optimum solution can be gained. In example 1, due to $\left|{-1.75}/{0.375}\right| \langle \left|{-2.25}/{0.25}\right|$, it fits the principle only when the row including x_1 is regards as the export equation. The principle is easy to understand both maximize and minimize problem, because it may make the object function value minish for maximize problem or augment for minimize problem. The principle fits to the example 2 and other examples that the author had done.

However in the practice, the author find that the exclusivity principle above have still exception such as the example 3. This is the reason that the paper is dissertated.

【Example 3】 Solving following integral programming using the cutting plane method

6 Xiong Yijie: "Solves ILP Cutting plane algorithm Astringent Question", "Operation and Management" 2003.2.

$$Max.Z = x_1 + x_2$$

$$\text{s.t.} \begin{cases} 2x_1 + x_2 \le 6 \\ 4x_1 + 5x_2 \le 20 \\ x_1, x_2 \ge 0 \ and \ x_1, x_2 \ as \ \text{int}eger \end{cases}$$

Solving the SLP problem, and getting the optimum simplex table such as table 5. Obviously, according to the exclusivity principle above, we should be selected the row including x_1 as the export equation. But actually, the cutting restriction that is created by the equation cannot obtain the optimum integer revolution using once iterative operation. Inverse, the optimum integer revolution can be obtained using the row including x_2 as the export equation through once iterative operation. The result is shown in Table 6. Obviously, the integer revolution is multi-revolution. And the other integer revolution is Z（2，2）=4.

Table 5 the optimum simplex table in example 3

		X_1 x_2 x_3 x_4
Z	13/3	0 0 -1/6 -1/6
X_1	5/3	1 0 5/6 -1/6
X_2	8/3	0 1 -2/3 1/3

Table 6 the simplex calculation table adding the cutting restriction

		X_1 x_2 x_3 x_4 x_5
Z	13/3	0 0 -1/6 -1/6 0
X_1	5/3	1 0 5/6 -1/6 0
X_2	8/3	0 1 -2/3 1/3 0
X_5	-2	0 0 （-1） -1 1
Z	-4	0 0 0 0 -1/6
X_1	0	1 0 0 -1 5/6
X_2	4	0 1 0 1 -2/3
X_3	2	0 0 1 1 -1

3. THE NEW METHOD TO CHOOSE CUTTING PLANE EQUATION

The above-mentioned circumstance explains that how to choose the fit cutting plane equation, still stay an experience observation stage, and don't resolve relevant material theories problem.

To resolve the fitting choice problem of cutting plane equation, we need to point out this problem why some cutting plane equation can find out the optimum integral solution quickly and effectively, and other can't? Researching to this problem, it isn't difficult for us to find that the cutting plane equation that can obtain optimum integral solution has stronger ability to cut the feasible field for the SLP than others. We might analyze the three examples above-mentioned.

We have already known that cutting constraint leaded from the second equation of in Table 1 is $x_1 \le 3$ (record for cutting constraint 2), but getting the integral part of cutting constraint from the first equation is $x_2 - x_4 - 1 \le 0$ (record for cutting constraint 1). The standard type constraint from example 1 as follow:

$$\begin{cases} 2x_1 + 3x_2 + x_3 = 12 \\ 2x_1 - x_2 + x_4 = 6 \end{cases}$$

To solve x_4 from the above-mentioned equation, and to go into cutting constraint 1, get $x_1 \le 3.5$. Obviously, the cutting constraint 2 has stronger sanction. Therefore, the cutting constraint 2 can get the optimum integral solution rapidly and availably.

For example 2, the two cutting constraints from Table 3 go into the following equations in example 2:

$$\begin{cases} x_3 = (3x_1 + x_2) - 4 \\ x_4 = (x_1 + 2x_2) - 4 \end{cases}$$

Get following equations:

$$\begin{cases} x_1 + x_2 \ge 6/5 \\ x_1 + x_2 \ge 11/5 \end{cases}$$

Obviously, for the minimum problem, the latter has stronger constraint force than the former.

For example 3, using the same method to get following two cutting constraints

$$\begin{cases} x_1 + x_2 \ge 21/5 \\ x_1 + x_2 \ge 4 \end{cases}$$

Three examples show the accuracy of our analytical conclusions. Is also say, the choosing principle of the cutting plane equation should be: to choose the cutting plane equation that has stronger constraint force to realize the iterative operation. But how to find the expert equation whose constraint force is stronger. Under the situation that the theories field hasn't provided the more valid method currently, the method above-mentioned should be commendable, only have some fussy. But this fussy is good for use relative to the low efficiency that uses the arbitrary cutting plane equation to choice the expert equation. Certainly, we should find out a kind of simple and easily method, but it is difficult because it need to explore in the practice, also need to break through in the theories of solving the ILP problem.

Here there are two points need to be pointed out. One is that using the cutting plane equation that has stronger sanction using the iterative operation is conditional. That is the cutting plane equation must have same inclined rate on the plane after using the method. Also the two variables must have same coefficient. Only so, we can determine elasticity of its constraint through the comparison of the different equation constant item. This, sometimes don't satisfy in the practice. As a result, the different cutting constraint that we get may not compare with each other. It is possible that the cutting plane restriction is unlike above-mentioned three examples in practice, which one is stronger, which one is poor, we cannot find at once. We can give following example.

【Example 4 】 Use to cut plane method to solve following integral programming

$$Max.Z = 3x_1 + 4x_2$$

$$\text{s.t.} \begin{cases} 3x_1 + 2x_2 \le 8 \\ x_1 + 5x_2 \le 9 \\ x_1, x_2 \ge 0 \; and \; x_1 \; and \; x_2 \; as \; \text{int} eger \end{cases}$$

The optimum simplex table shown in Table 7 can be obtained from the equations.

Table 7 the optimum simplex table in example 4

		X_1	x_2	x_3	x_4
Z	142/13	0	0	-11/13	-6/13
X_1	22/13	1	0	5/13	-2/13
X_2	19/13	0	1	-1/13	3/13

According to above-mentioned optimum simplex table, the two cutting constraints are:

$$\begin{cases} x_1 - x_4 - 1 = \frac{9}{13} - (\frac{5}{13}x_3 + \frac{11}{13}x_4) \le 0 \\ x_2 - x_3 - 1 = \frac{6}{13} - (\frac{12}{13}x_3 + \frac{3}{13}x_4) \le 0 \end{cases}$$

Adding the x_3 and x_4 expression in the standard type stipulation of above-mentioned, getting:

$$\begin{cases} 2x_1 + 5x_2 \le 10 \\ x_1 + x_2 \le 3 \end{cases}$$

Like this kind of situation, it is difficult for us to judge whose restriction is stronger. That needs us to recur to the plane chart to judge. Feasible field and cutting restriction in example 4 is shown in Fig 2. Obviously, the top point of the cutting restriction 2 contains an integer point (2,1). The cutting restriction 2 deletes the red part of feasible field. And the cutting restriction 1 deleting the blue part and red part don't contain any integer points.

The other problem needing to discus is how to use the cutting plane to make high efficiency, such as example 4，if taking the cutting restriction 2 into Table 7, it cannot obtain the optimum resolution through once iterative operation. If transfiguration the cutting restriction as：$\frac{6}{13} + \frac{1}{13}x_3 - \frac{3}{13}x_4 \le 0$，it can obtain optimum integer value (2,1) expediently through once iterative operation. However it is can't be explained why make it great little and equal 0.

So the principle in this paper is only attempt, the study of the cutting plane equation need to break through in theory.

THE AUTHOR

Xiong Yijie, Business Administration School, Xi'an University of Technology, Xi'an, Shanxi, 710048, P.R. China.

E-mail: xiongyj@xaut.edu.cn

Ren Jing, Business Administration School, Xi'an University of Technology, Xi'an, Shanxi, 710048, P.R. China.